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DIFFERENTIAL EQUATIONS IN AIRPLANE MECHANICS.

By

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For determining the motion of an airplane, we have adopted the hypothesis that the reactions of the air depend entirely on the relative speed of the airplane. Even if we adopt the simplest laws of resistance, we obtain differential equations which we can not integrate explicitly. If we confine ourselves to the motion in a vertical plane and, at the same time, assume a constant angle of attack, we still obtain differential equations which can not be integrated by elementary methods.

In the following paragraphs, we will first draw some conclusions of purely theoretical interest, from the general equations of motion. At the end, we will consider the motion of an airplane, with the engine dead and with the assumption that the angle of attack remains constant. Thus we arrive at a simple result, which can be rendered practically utilizable for determining the trajectory of an airplane descending at a constant steering angle.

Let us assume that the airplane moves in its plane of symmetry, considered vertical. Let x and y represent a system of coordinates in this plane, the axis of the x lines being horizontal and the axis of the y lines being vertical and upward. Let us designate by u and v the projections of the velocity w

* From "La Technique Aeronautique," May, 1921.

of the center of gravity of the airplane on the axes of x and y and by the angle ψ the angle of the velocity with the axis of the x lines. Let us designate by M the mass of the airplane and by MI its moment of inertia with relation to an axis passing through the center of gravity at right angles to the plane xy. The forces acting on the airplane are:

Its weight Mg directed toward $-y$:

Propeller thrust T. of which we will designate by MA and MB the components in the direction of w and at right angles to w;

Drag, $M\,a\,w^2$, in the direction $-w$;

Lift, $M\,b\,w^2$, at right angles to w.

Under these conditions the theory of the motion of the center of gravity gives us

$$M\frac{du}{dt} = -M\,a\,w^2 \cos\psi + M\,A\cos\psi - M\,b\,w^2 \sin\psi - M\,B\sin\psi.$$

$$M\frac{dv}{dt} = -M\,a\,w^2 \sin\psi + M\,A\sin\psi + M\,b\,w^2\cos\psi + M\,B\cos\psi - Mg\,dt\,.$$

By substituting

$$\cos\psi = \frac{u}{w} \qquad \sin\psi = \frac{v}{w}$$

we have

$$(1) \quad \frac{du}{dt} = -w\,(a\,u + b\,v) + \frac{Au - Bv}{w}$$

$$(2) \quad \frac{dv}{dt} = -w\,(a\,v - b\,u) + \frac{Av + Bu}{w} - g$$

$$w = \sqrt{u^2 + v^2}\,.$$

If we project these forces on the tangent and on the perpendicular to the trajectory, we obtain

(3) $\frac{dw}{dt} = - aw^2 - g \sin \psi + A.$

(4) $w\frac{d\psi}{dt} = bw^2 - g \cos \psi + B.$

By letting

$$a = \rho \sin \alpha \qquad A = K \cos \beta$$

$$b = \rho \cos \alpha \qquad B = K \sin \beta$$

We can, by introducing imaginary quantities, combine equations (1) and (2) into a single equation

(5) $\frac{d(u + iv)}{dt} = \left(i\, w\, \rho\, e^{i\alpha} + \frac{Ke^{i\beta}}{w}\right)(u + iv) - gi.$

The moment of the air resistances about the center of gravity can be expressed in the form

$$M\, w^2\, G(\varphi) - M\, w\frac{d(\varphi + \psi)}{dt}\; H(\varphi).$$

in which G and H are the periodic functions of the angle of attack φ, which varies with the steering angle. We then have, according to the theory of motion about the center of gravity,

(6) $I\, \frac{d^2\, (\varphi + \psi)}{dt^2} = w^2\, G(\varphi) - w\, \frac{d(\varphi + \psi)}{dt}\, H(\varphi).$

Let us assume that the influence of altitude may be neglected. By multiplying equation (1) by u and equation (2) by v and adding the results, we obtain

$$(7)\quad \frac{1}{2}\,\frac{dw^2}{dt} = - aw^3 + Aw - gv.$$

The quantity a, necessarily implying a lower positive limit a', is obtained from equation (7).

$$(8)\quad \frac{1}{2}\,\frac{dw^2}{dt} < - w\left(a'w^2 - (P + g)\right).$$

MP representing the maximum propeller thrust, from which we reason that the speed can not go on increasing indefinitely. To be more exact, it must remain below

$$\sqrt{\frac{P + g}{a'}}$$

in which a'' is any quantity smaller than a'.

By means of equation (1) we can easily demonstrate the following theorem, which is practically self-evident. _If the engine is dead, it is not possible by any maneuver of the elevator to keep the airplane above any fixed horizontal line for any arbitrary period of time_. We find, in fact, by integrating equation (7) between t_o and t (A and B being zero)

$$\frac{1}{2}\left(w_t{}^2 - w_o{}^2\right) = - \int_{t_o}^{t} aw^3\,dt - g\,(y_t - y_o).$$

Taking into account the inequality $a > a' > 0$, the hypothesis $y_t > 0$ gives us

$$a' \int_{t_o}^{t} w^3\,dt < \frac{1}{2}\,w_o{}^2 + gy_o.$$

It follows that the integral

$$\int_{t_o}^{\infty} w^3\,dt$$

is convergent. Remarking that by reason of equations (7) and (8), $\frac{dw^2}{dt}$ is limited, we reason from this that w tends towards zero, when t tends toward infinity. Hence we have also

$$(9) \quad \lim_{t=\infty} v = 0, \qquad \lim_{t=\infty} u = 0$$

Consequently, there is an infinity of values of t, t_1, t_2 ... t_n such that we have

$$(10) \quad \lim_{n=\infty} \left(\frac{dv}{dt}\right)_{t_n} = 0, \qquad \lim_{n=\infty} \left(\frac{du}{dt}\right)_{t_n} = 0, \qquad \lim_{t=\infty} t_n = \infty$$

By substituting $t = t_n$ in the equation

$$\frac{dv}{dt} = - w\ (av - bu) - g$$

and by making n tend toward infinity, we arrive at the absurdity $g = 0$. The hypothesis $y_t > 0$ is therefore inadmissible.

Let us now assume the thrust and angle of attack to remain constant. On dividing equation (6) by w^2, we see that for a constant steering angle of the elevator, this hypothesis concerning the angle of attack is judtified in proportion as I and H are smaller and w is larger. From equations (1) and (3) we deduce the differential equation

$$(11) \quad \frac{du}{(u^2 + v^2)\ (au + bv) - Au + Bv} =$$

$$\frac{dv}{(u^2 + v^2)\ (av - bu) - Av - Bu + g\ \sqrt{u^2 + v^2}}$$

in which a, b, A and B are constants. Once this equation is integrated, t, x and y are obtained by quadratures. We can accordingly confine ourselves to the study of the integral curves of equation (11), that is, to determining the hodograph of the motion. We can apply to equation (11) the methods given by Mr. Poincaré in his memoirs: "On curves defined by a differential equation" (Journal de Mathématiques pures et appliquées, 1881-1886). We see that equations (1) and (2) define u and v for every value of t, when the initial velocity $u_0 v_0$ is given. On considering u and v as cartesian coordinates of a point in the plane, the point (u,v) describes a certain characteristic curve, when t varies from 0 to infinity. We call "singular points" those where the second members of equations (1) and (2) disappear at the same time or become discontinuous. In the case under consideration, there is, aside from the origin, only one other singular point at finite distance. We have already seen (inequality 8) that the characteristics remain at a finite distance. Under this condition, the theory of Poincaré shows us that a knowledge of the singular points and closed characteristics (limited cycles) at a finite distance suffices for finding the course of the integral curves, when t tends toward infinity.

Instead of going deeply into this study here, we shall take up by a simpler method, the case where the propeller thrust is zero. Equation (11) is reduced to

$$\text{(12)} \qquad \frac{du}{w\,(au + bv)} = \frac{dv}{w\,(av - bu) + g}$$

For $\alpha = 0$,* that is to say, $a = 0$, $b = \rho$, this equation is readily integrated by means of the elementary functions. We have, in fact,

$$\frac{du}{\rho\, vw} = \frac{dv}{-\rho\, uw + g} = \frac{w\, d\, w}{gv}$$

Hence

$$g du = \rho\, w^2\, dw$$

$$\rho\, \frac{w^3}{3} - gu = C \qquad (C = \text{arbitrary constant})$$

By varying C we obtain a system S of closed, non-intersecting curves, which embrace the whole plane.

In the case $\alpha = \pi/2$, corresponding to the motion of a sphere in a resisting medium, equation (12) can be integrated by quadratures, as demonstrated by Liouville.

Although we do not know for α any general integral of equation (12), we can construct its integral curves by a simple method. We have, in fact, the following theorem. The integral curves of equation (12) are the trajectories of the angle α of the system of S curves.

$$\rho\, \frac{(u^2 + v^2)^{3/2}}{3} - gu = \text{constant}$$

provided we consider these trajectories with reference to the rec-

* In his article "Le vol Aérien" (Aerial Flight), Mr. Lanchester discussed the fall of an airplane under this hypothesis, which returns to the assumption that the airplane does not lose energy.

tangular axes which make the angle α (in the positive direction) with the original trajectories (see figure).

For the demonstration, we write the differential equations in the following condensed form

$$\frac{d\,(u + iv)}{dt} = iw\,\rho e^{i\alpha}\,(u + iv) - gi \tag{13}$$

which is obtained from equation (5) by letting $K = 0$. By making $\alpha = 0$ in equation (13) we find, for the S system, the differential equation

$$\frac{d\,(u + iv)}{dt} = iw\,\rho\,(u + iv) - gi. \tag{14}$$

We obtain the differential equation of the trajectories of the angle α by multiplying the second member of equation (14) by $e^{i\alpha}$. Hence

$$\frac{d\,(u + iv)}{dt} = iw\,\rho\, e^{i\alpha}\,(u + iv) - gie^{i\alpha} \tag{15}$$

The change of axes indicated in the above statement is obtained by multiplying $u + iv$ in equation (15) by $e^{i\alpha}$. By dividing the ratio thus obtained by $e^{i\alpha}$, we return to equation (13), which demonstrates the theorem.

All the characteristics tend toward the point

$$u = \sqrt{\frac{g}{\rho}}\cos\alpha, \qquad v = -\sqrt{\frac{g}{\rho}}\sin\alpha$$

which corresponds to a rectilinear and uniform descent.* If the in-

* If an integral curve passes near the point $u = v = 0$, we can hardly expect that it will correspond to the real motion of the airplane, because the hypothesis that the angle of attack remains constant is not justified for low speeds.

tegral curve cuts the axis of the negative u's, n times, the airplane executes n loops. We come, therefore, to the following conclusion. Whatever be the initial velocity, the airplane, after executing, if necessary, a finite number of loops, acquires a motion which approaches indefinitely a state of rectilinear and uniform descent.

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